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Parametrically Driven Pendulum and Exact Solutions

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The parametrically driven pendulum $\ddot{x}+f_1(t)\dot{x}+f_2(t)\sin x=0$ cannot be solved in closed form for arbitrary functions f_1, f_2. We apply the Painlevé test to obtain the constraint on the functions f_1 and f_2 for which the equation passes the test. The constraint on f_1 and f_2, a differential equation which f_1 and f_2 obey, is discussed and solutions are given. The third Painlevé transcendent plays a central role.

For nonlinear ordinary and partial differential equations the general solution usually cannot be given explicitly. It is desirable to have an approach to find out whether a given nonlinear differential equation can be solved explicitly. We investigate the parametrically driven pendulum

$$\ddot{x}+f_1(t)\dot{x}+f_2(t)\sin x=0 \tag{1}$$

where f_1 and f_2 are smooth functions. If $f_1(t)=0$ and $f_2(t)=1$, then (1) reduces to the equation of the pendulum $\ddot{x}+\sin x=0$. This equation can be solved in terms of elliptic functions. For arbitrary functions f_1 and f_2 the nonlinear equation (1) cannot be solved explicitly. For example, if $f_1(t)=c_1$ and $f_2(t)=1+c_2\sin(\Omega t)$, where c_1 and c_2 are constants, then (1) can show chaotic behavior (Herbst and Steeb, 1988) for certain values of c_1, c_2, and Ω. Obviously, in this case (1) cannot be integrated.

We would like to find the constraint on the functions f_1 and f_2 such that (1) can be solved or reduced to one of the Painlevé transcendents. We apply the Painlevé analysis (Steeb and Euler, 1988, and references therein) to find the condition on f_1 and f_2. A similar program was performed for the damped anharmonic oscillator (Euler *et al.*, 1989; Duarte *et al.*, 1990).

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In order to perform the Painlevé analysis, we have to apply the transformation $v(t)=\exp[ix(t)]$. Then (1) takes the form

$$vv''-v'^2+f_1(t)vv'+\tfrac{1}{2}f_2(t)(v^3-v)=0 \tag{2}$$

Equation (2) is considered in the complex domain. Inserting the Laurent expansion

$$v(t)=\sum_{j=0}^{\infty} a_j(t-t_1)^{j-n} \tag{3}$$

we obtain $n=2$. At the resonance $r=2$ we obtain the condition

$$f_2''f_2-f_2'^2+f_2'f_2f_1+2f_1'f_2^2+2f_1^2f_2^2=0 \tag{4}$$

If this condition is satisfied, the expansion coefficient a_2 is arbitrary and (2) passes the Painlevé test. A remark is in order for applying the Painlevé test for nonautonomous systems. The coefficients that depend on the independent variable must themselves be expanded in terms of $t-t_1$, where $t\equiv(t-t_1)+t_1$. If nonautonomous terms enter the equation at lower order than the dominant balance, the above-mentioned expansion turns out to be unnecessary, whereas if the nonautonomous terms are at the dominant balance level, they must be expanded with respect to $t-t_1$. Obviously, in our case the functions f_1 and f_2 do not enter the expansion at the dominant level.

A necessary condition that an ordinary differential equation is of Painlevé type is that it passes the Painlevé test. We say that an ordinary differential equation is of Painlevé type if all its solutions possess the Painlevé property, i.e., their only singularities are poles or nonmovable critical points.

We discuss (4) by considering various cases.

Case I. Let $f_1(t)=0$. Then we have $f_2''f_2-f_2'^2=0$. The general solution is given by $f_2(t)=c_1\, e^{c_2 t}$, where c_1 and c_2 are constants of integration. Obviously this equation is of Painlevé type. Inserting this solution into (2) yields

$$vv''-v'^2+\tfrac{1}{2}c_1\, e^{c_2 t}(v^3-v)=0 \tag{5}$$

This equation is a special case of ($c_2=1$)

$$vv''-v'^2-e^t(\alpha v^3+\beta v)-e^{2t}(\gamma v^4+\delta)=0 \tag{6}$$

where α, β, γ, and δ are complex constants. Equation (6) is the third Painlevé transcendent after a transformation of the independent variable (Ince, 1956). It is well known that this equation is not integrable in terms of classical transcendents.

Case II. Let $f_2(t)=0$. Then (4) is satisfied and (2) passes the Painlevé test. Obviously (1) is now linear.

Case III. Let $f_1(t)=f_2(t)\equiv f(t)$. Then we have

$$f''f-f'^2+3f'f^2+2f^4=0 \tag{7}$$

We now perform the Painlevé test for (7) by inserting the Laurent expansion (3). We obtain $n=1$. All terms are dominant. There are two branches, namely $a_0=1$ and $a_0=1/2$. Let us first discuss the branch with $a_0=1$. The only resonance for this branch is $r=-1$ (twofold). Since all terms are dominant, we obtain a particular solution to (7) for the first branch, namely $f(t)=1/t$. Inserting this solution into (2) yields

$$vv''-v'^2+\frac{1}{t}vv'+\frac{1}{2t}(v^3-v)=0 \tag{8}$$

This is a special case of the third Painlevé transcendent. For the second branch the resonances are $r=-1$ and $r=1/2$. Inserting the expansion

$$v(t)=\sum_{j=0}^{\infty} a_j(t-t_1)^{j/2-1} \tag{9}$$

into (7), we find that the expansion coefficient at $r=1/2$ can be chosen arbitrarily. Thus, (7) does not pass the Painlevé test, but the so-called weak Painlevé test (Steeb and Euler, 1988, and references therein). Since all terms are dominant, we find again a particular solution, namely $f(t)=1/(2t)$. Inserting this solution into (2) leads again to a special case of the third Painlevé transcendent.

Case IV. Let $f_1(t)=c$ and $f_2(t)=f(t)$, where c is a nonzero constant. Then (4) takes the form

$$f''f-f'^2+cff'+2c^2f^2=0 \tag{10}$$

The general solution to (10) is given by

$$f(t)=\exp\left[-\frac{K_1}{c}(e^{-ct}-1)-2ct+K_2\right] \tag{11}$$

where K_1 and K_2 are the constants of integration. Obviously, (10) has the Painlevé property.

Case V. Let f_2 be a given smooth function. Then the condition (4) is a Riccati equation. We find

$$f_1'=-f_1^2+gf_1-g' \tag{12}$$

where $2g=f_2'/f_2$. We obtain the general solution of (12) as

$$f_1(t)=\frac{d}{dt}\ln\left\{\frac{1}{P(t)}\left[c_1\int^t P(s)\,ds+c_2\right]\right\} \tag{13}$$

where

$$P(t)=\exp\left[\int^{t} g(s)\,ds\right] \tag{14}$$

We recall that the 50 ordinary differential equations of second order of Painlevé type are just representatives of equivalence classes. The group under which the classification is done is given by

$$T=\phi(t)$$
$$X(T(t))=\frac{\psi_1(t)x(t)+\psi_2(t)}{\psi_3(t)x(t)+\psi_4(t)} \tag{15}$$

where ψ_1, ψ_2, ψ_3, ψ_4, and ϕ are analytic functions of t. We conjecture that (2) together with f_1 given by (13) and arbitrary f_2 can be transformed to the third Painlevé transcendent.

For certain choices of the parameters, the Painlevé transcendents II–V admit one-parameter families of solutions expressible in terms of classical transcendental functions, such as Airy, Bessel, Weber–Hermite, and Whittaker, respectively (Gromak, 1978).

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